Online Model Appendix:

Lakes and Economic Development:
Evidence from the Permanent Shrinking of Lake Chad*

Remi Jedwab Federico Haslop Román D. Zárate Carlos Rodríguez-Castelan

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A Definition of Sequential and Counterfactual Equilibrium

We define the sequential and counterfactual equilibrium based on Caliendo et al. (2019) (henceforth, CDP). Let's define a change of any variable as $\dot{x}_t = \frac{x_t}{x_{t-1}}$ and $u_t^{nj} = \exp(V_t^{nj})$.

Temporary equilibrium: Given $(L_t, A_t, \kappa_t, K, \Gamma)$, where Γ is the vector of all parameters. A temporary equilibrium in period t is defined as a vector of wages $w_t = w(L_t, A_t, \kappa_t, K, \Gamma)$ that solves equations (8) to (13) in the main text.

Sequential equilibrium: Given an initial allocation $(L_0, \{A_t, \kappa_t\}_{t=0}^{\infty}, K, \Gamma)$ a *sequential competitive equilibrium* of the model is a vector of $\{L_t, \mu_t, V_t, w_t(L_t, A_t, \kappa_t, K, \Gamma)\}_{t=0}^{\infty}$ that solves the dynamic equations (4) to (6) and the temporary equilibrium.

We solve the sequential equilibrium using propositions 1 and 2 from CDP, incorporating land into the model. These propositions state that given an initial allocation of the economy $(L_0, K_0, \pi_0, X_0, \mu_{-1})$ and an anticipated sequence of changes in fundamentals $\{\dot{A}_t, \dot{\kappa}_t\}_{t=0}^{\infty}$, we can find the sequential competitive equilibrium without knowing the level of fundamentals by solving the following system of equations using the initial data on the labor mobility matrix and initial trade flows across locations for each sector:

$$\mu_t^{nj,ik} = \frac{\mu_{t-1}^{nj,ik}(\dot{u}_{t+1}^{ik})^{\frac{\beta}{\nu}}}{\sum_{r=1}^{J}\sum_{m=1}^{N}\mu_{t-1}^{nj,rm}(\dot{u}_{t+1}^{rm})^{\frac{\beta}{\nu}}} \tag{A.1}$$

$$\dot{u}_t^{nj} = \frac{\dot{w}_t^{nj}}{\dot{P}_t^{nj}}\left(\sum_{r=1}^{J}\sum_{m=1}^{N}\mu_{t-1}^{nj,rm}(\dot{u}_{t+1}^{rm})^{\frac{\beta}{\nu}}\right)^{\nu} \tag{A.2}$$

$$L_{t+1}^{nj} = \sum_{k=1}^{J}\sum_{i=1}^{N}\mu_t^{nj,ik}L_t^{ik} \tag{A.3}$$

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where $\dot{w}_{t+1}^{nj}$ and $\dot{P}_{t+1}^{nj}$ solve the the temporary equilibrium at period $t+1$, which corresponds to the following system of equations:

$$\dot{x}_{t+1}^{nj} = (\dot{w}_{t+1}^{nj})^{\phi^{nj}\eta^{nj}}(\dot{r}_{t+1}^{n})^{\phi^{nj}(1-\eta^{nj})}\prod_{s=1}^{J}\left(\dot{P}_{t+1}^{ns}\right)^{\phi^{nj,ns}} \tag{A.4}$$

$$\dot{P}_{t+1}^{nj} = \left(\sum_{m=1}^{N}\pi_t^{nj,mj}(\dot{x}_{t+1}^{mj}\dot{\kappa}_{t+1}^{nj,mj})^{-\theta^j}(\dot{A}_{t+1}^{mj})^{\theta^j}\right)^{\frac{-1}{\theta^j}} \tag{A.5}$$

$$\dot{\pi}_{t+1}^{nj,ij} = \frac{(\dot{x}_{t+1}^{ij}\dot{\kappa}_{t+1}^{nj,ij})^{-\theta^j}(\dot{A}_{t+1}^{ij})^{\theta^j}}{\sum_{m=1}^{N}\pi_t^{nj,mj}(\dot{x}_{t+1}^{mj}\dot{\kappa}_{t+1}^{nj,mj})^{-\theta^j}(\dot{A}_{t+1}^{mj})^{\theta^j}} \tag{A.6}$$

$$X_{t+1}^{nj} = \sum_{s=1}^{J}\sum_{i=1}^{N}\phi^{ns,nj}\pi_{t+1}^{is,ns}X_{t+1}^{is} + \alpha^{nj}\left(\sum_{k=1}^{J}\dot{w}_{t+1}^{nk}\dot{L}_{t+1}^{nk}w_t^{nk}L_t^{nk} + \dot{r}_{t+1}^{n}r_t^{n}K^{n}\right) \tag{A.7}$$

$$\dot{w}_{t+1}^{nj}\dot{L}_{t+1}^{nj} = \frac{\phi^{nj}\eta^{nj}\sum_{i=1}^{N}\pi_{t+1}^{ij,nj}X_{t+1}^{ij}}{w_t^{nj}L_t^{nj}} \tag{A.8}$$

$$\dot{r}_{t+1}^{n} = \sum_{j=1}^{J}\left(\frac{1-\eta^{nj}}{\eta^{nj}}\right)\frac{\dot{w}_{t+1}^{nj}\dot{L}_{t+1}^{nj}w_t^{nj}L_t^{nj}}{r_t^{n}K^{n}} \tag{A.9}$$

Counterfactual equilibrium: The last concept to define is the counterfactual equilibrium. As in CDP, to compute counterfactuals, we assume that agents do not know the shock at $t = 0$, but learn about the change in the path of fundamentals at $t = 1$. Let's define $\hat{y}_{t+1} = \dot{y}'_{t+1}/\dot{y}_{t+1}$ as the ratio of time changes between the counterfactual equilibrium and the initial baseline equilibrium for any variable y, where $\dot{y}'_{t+1} = \frac{y'_{t+1}}{y'_t}$ and y'_{t+1} corresponds to the level of y at time $t+1$ in the new path. This means that if $\hat{y}_{t+1} \neq 1$, there is a change in variable y in the new equilibrium. Following proposition 3 from CDP, we can solve for the counterfactual equilibrium without information on the baseline fundamentals and using the initial sequential equilibrium by solving the following system of equations:

$$\mu_t^{'nj,ik} = \frac{\mu_{t-1}^{'nj,ik}\dot{\mu}_t^{nj,ik}(\hat{u}_{t+1}^{ik})^{\frac{\beta}{\nu}}}{\sum_{r=1}^{J}\sum_{m=1}^{N}\mu_{t-1}^{'nj,rm}\dot{\mu}_t^{nj,rm}(\hat{u}_{t+1}^{rm})^{\frac{\beta}{\nu}}} \tag{A.10}$$

$$\hat{u}_t^{nj} = \frac{\hat{w}_t^{nj}}{\hat{P}_t^{nj}}\left(\sum_{r=1}^{J}\sum_{m=1}^{N}\mu_{t-1}^{'nj,rm}\dot{\mu}_t^{nj,rm}(\hat{u}_{t+1}^{rm})^{\frac{\beta}{\nu}}\right)^{\nu} \tag{A.11}$$

$$L_{t+1}^{'nj} = \sum_{k=1}^{J}\sum_{i=1}^{N}\mu_t^{'nj,ik}L_t^{ik} \tag{A.12}$$

for all j, n, i, and k; where $\{\hat{w}_{t+1}^{nj}\}_{t=0}^{\infty}$ and $\{\hat{P}_{t+1}^{nj}\}_{t=0}^{\infty}$ solve the temporary equilibrium at each period $t+1$. In particular, they solve the following system of equations:

$$\hat{x}_{t+1}^{nj} = (\hat{w}_{t+1}^{nj})^{\phi^{nj}\eta^{nj}}(\hat{r}_{t+1}^{n})^{\phi^{nj}(1-\eta^{nj})}\prod_{s=1}^{J}\left(\hat{P}_{t+1}^{ns}\right)^{\phi^{nj,ns}} \tag{A.13}$$

$$\hat{P}_{t+1}^{nj} = \left(\sum_{m=1}^{N} \pi_t^{'nj,mj} \dot{\pi}_{t+1}^{nj,mj} (\hat{x}_{t+1}^{mj} \hat{\kappa}_{t+1}^{nj,mj})^{-\theta^j} (\hat{A}_{t+1}^{mj})^{\theta^j} \right)^{\frac{-1}{\theta^j}} \tag{A.14}$$

$$\pi_{t+1}^{'nj,ij} = \frac{\pi_t^{'nj,ij} \dot{\pi}_{t+1}^{nj,ij} (\hat{x}_{t+1}^{ij} \hat{\kappa}_{t+1}^{nj,ij})^{-\theta^j} (\hat{A}_{t+1}^{ij})^{\theta^j}}{\sum_{m=1}^{N} \pi_t^{'nj,mj} \dot{\pi}_{t+1}^{nj,mj} (\hat{x}_{t+1}^{mj} \hat{\kappa}_{t+1}^{nj,mj})^{-\theta^j} (\hat{A}_{t+1}^{mj})^{\theta^j}} \tag{A.15}$$

$$X_{t+1}^{'nj} = \sum_{s=1}^{J} \sum_{i=1}^{N} \phi^{ns,nj} \pi_{t+1}^{'is,ns} X_{t+1}^{'is} + \alpha^{nj} \left(\sum_{k=1}^{J} \hat{w}_{t+1}^{nk} \hat{L}_{t+1}^{nk} \dot{w}_{t+1}^{nk} \dot{L}_{t+1}^{nk} w_t^{'nk} L_t^{'nk} + \hat{r}_{t+1}^{n} \dot{r}_{t+1}^{n} r_t^{'n} K^n \right) \tag{A.16}$$

$$\hat{w}_{t+1}^{nj} \hat{L}_{t+1}^{nj} = \frac{\phi^{nj} \eta^{nj} \sum_{i=1}^{N} \pi_{t+1}^{'nj,ij} X_{t+1}^{'ij}}{w_t^{'nj} L_t^{'nj} \dot{w}_{t+1}^{nj} \dot{L}_{t+1}^{nj}} \tag{A.17}$$

$$\hat{r}_{t+1}^{n} = \sum_{j=1}^{J} \left(\frac{1-\eta^{nj}}{\eta^{nj}} \right) \frac{\hat{w}_{t+1}^{nj} \hat{L}_{t+1}^{nj} \dot{w}_{t+1}^{nj} \dot{L}_{t+1}^{nj} w_t^{'nj} L_t^{'nj}}{r_t^{'n} \dot{r}_{t+1}^{n} K^n} \tag{A.18}$$

Then, we can use Proposition 3 from CDP to solve for the counterfactual equilibrium.

A.1 Algorithm

We model the Lake Chad shrinkage by solving the sequential equilibrium. We use the following contraction mapping based on CDP and Rodriguez-Clare et al. (ming) to find the solution for the economy. At t=0, agents know the evolution of the productivity path and trade costs. Let's denote $(\dot{u}_t^{nj})^s$ as simulation s for the vector of utilities, $(\dot{w}_t^{nj})^r$ as simulation r for the wage vector, and $(\dot{P}_t^{nj})^v$ as simulation v for the price indices.

1. Start with an initial vector of utilities $\{(\dot{u}_t^{nj})^0\}_{t=0}^{\infty}$.

2. Construct the new migration flows matrix using equation A.1 and the initial migration matrix.

3. Construct employment in each sector and location using equation A.3.

4. In an inner loop, solve for the temporary equilibrium given the path of productivity changes $\dot{A}_{t+1}^{nj}$ and the data from the previous period t:

   - Start with an initial vector of wages $(\dot{w}_{t+1}^{nj})^0$
   - Solve for the change in the land rate given the change in wages and the data from period t using equation A.9.
   - In an inner loop, solve for the vector of price indices $\dot{P}_{t+1}^{nj}$:
     - Solve $\dot{x}_{t+1}$ using equation A.4
     - Construct the change between $t+1$ and t for the price index using the trade shares from the previous period and equation A.5. Denote this new price index as $(\dot{P}_{t+1}^{nj})'$
     - Update a new vector of price indices using some convergence parameter ζ and the price index from simulation v:

$$(\dot{P}_{t+1}^{nj})^{v+1} = \zeta(\dot{P}_{t+1}^{nj})' + (1-\zeta)(\dot{P}_{t+1}^{nj})^{v}$$

- Continue the algorithm until the difference between simulation v and $v+1$ in the price indices is lower than some tolerance factor tol.

- With the updated $\dot{x}_{t+1}$, construct the new trade shares using equation A.6.
- Solve for the expenditure in each sector and location in period $t+1$ using the data from period t and equation A.7.
- Construct a new vector of wages $(\dot{w}_{t+1}^{nj})'$ using the trade shares and total expenditure from period $t+1$, and equation A.8.
- Update a new vector of wages using some convergence parameter ν and the wages from simulation r:

$$(\dot{w}_{t+1}^{nj})^{r+1} = \nu(\dot{w}_{t+1}^{nj})' + (1-\nu)(\dot{w}_{t+1}^{nj})^{r}$$

- Continue the algorithm until the difference between simulation r and $r+1$ is lower than some tolerance factor tol.

5. Construct a new vector of utilities $(\dot{u}_{t+1}^{nj})'$ recursively using equation A.2.
6. Update a new vector of utilities for simulation $s+1$ with some convergence parameter ρ:

$$(\dot{u}_{t+1}^{nj})^{s+1} = \rho(\dot{u}_{t+1}^{nj})' + (1-\rho)(\dot{u}_{t+1}^{nj})^{s}$$

7. Continue the algorithm until the difference in the change in utilities over time between simulation s and $s+1$ is lower than some tolerance factor tol.

B Model inversion to recover initial trade flows

The model requires knowing initial trade flows to solve for the sequential and counterfactual equilibrium. Unfortunately, such data is not available at the subdistrict level in Sub-Saharan Africa. Hence, we follow Allen and Arkolakis (2025) and we recover trade flows in the initial period $t=0$ by inverting the model under the non-trade deficit condition. In particular, knowing the distribution of trade costs $(\kappa^{nj,ij})$, total revenue in each sector and region (Y^{nj}), total employment (L^{nj}), total land (K^{n}), expenditure shares in the production function (β^{nj} and $\phi^{nj,nk}$), expenditure shares in consumption across sectors (α^{nj}), and the trade elasticity (θ^{j}), we can recover the productivity vector A^{nj} in the initial period by solving the following system of equations.[1]

$$X^{nj} = \sum_{k=1}^{J} \phi^{nj,nk} \underbrace{\sum_{i=1}^{N} \frac{(x^{nk}\kappa^{ik,nk})^{-\theta^k}(A^{nk})^{\theta^k}}{\sum_{m=1}^{N}(x^{mk}\kappa^{ik,mk})^{-\theta^k}(A^{nk})^{\theta^k}} X^{ik}}_{Y^{nk}\text{: Gross production in the initial period } nk} + \alpha^{nj} \underbrace{\left(\sum_{k=1}^{J} w^{nk}L^{nk} + r^n K^n\right)}_{\text{Final consumption with no deficits}}, \quad \text{(B.1)}$$

where

$$x^{nj} = (w^{nj})^{\phi^{nj}\eta^{nj}}(r^{nj})^{\phi^{nj}(1-\eta^{nj})}\prod_{s=1}^{J}(P^{ns})^{\phi^{nj,ns}}.$$

To construct the vector of wages and land prices, we use the total wage bill and land rents divided by total employment and land in region n:

$$w^{nj} = \frac{\eta^{nj}\phi^{nj}Y^{nj}}{L^{nj}} \quad \text{(B.2)}$$

$$r^{n} = \frac{1}{K^n}\left(\sum_{j=1}^{J}(1-\eta^{nj})\phi^{nj}Y^{nj}\right). \quad \text{(B.3)}$$

The system of equations B.1 has NJ equations and NJ unknowns, and we solve it following the Alvarez and Lucas algorithm that is described in the next subsection.

After we recover the productivity vector A^{nj}, we can construct trade flows within each sector by solving:

$$\pi^{nj,ij} = \frac{(x^{ij}\kappa^{nj,ij})^{-\theta^j}(A^{ij})^{\theta^j}}{\sum_{m=1}^{N}(x^{mj}\kappa^{nj,mj})^{-\theta^j}(A^{mj})^{\theta^j}} \quad \text{(B.4)}$$

We then use these initial trade flows to solve for the counterfactual and sequential equilibrium.

B.1 Algorithm

The algorithm works in the following way. Recall that we have data at the subdistrict and sector level on: i) trade costs across locations; ii) total revenue in each sector and location, iii) total employment and total land, and iv) expenditure shares in the production function.

1. We construct the expenditure shares in the consumption basket by assuming that all locations in the four countries have the same expenditure shares:

$$\alpha^{j} = \frac{\sum_{n=1}^{N} GDP^{nj}}{\sum_{k=1}^{J}\sum_{n=1}^{N} GDP^{nk}},$$

[1] We omit the time subindex since we do this only for the initial period in the cross-section.

where GDP^{nj} is the total revenue of location n and sector j without including the expenditure in intermediate inputs.

2. Solve for the wage and land price using equations B.2 and B.3.

3. With data on total revenue Y^{nj}, total wage bill $w^{nj}L^{nj}$, total land rents $K^n r^n$, and α^j construct the total expenditure of location n in sector j:

$$X^{nj} = \sum_{k=1}^{J} \phi^{nk,nj} Y^{nj} + \alpha^j \left(\sum_{k=1}^{J} w^{nk} L^{nk} + r^n K^n \right)$$

4. Assume an initial vector of productivity $(A^{nj})^0$ and denote a simulation with the upper index s. This means that the vector $(A^{nj})^s$ corresponds to the simulation s in the algorithm.

5. In an inner loop, solve for the price index in each location and sector given the vector $(A^{nj})^s$, w^{nj}, and r^n:

   - Assume an initial vector of price indices $(P^{nj})^0$ and denote simulation r by $(P^{nj})^r$.
   - Construct the variable x^{nj}

   $$(x^{nj})^r = (w^{nj})^{\phi^{nj}\eta^{nj}} (r^{nj})^{\phi^{nj}(1-\eta^{nj})} \prod_{k=1}^{J} ((P^{nk})^r)^{\phi^{nj,nk}}.$$

   - Construct a new price index for each location and sector $(P^{nj})'$ based on the CES formula of price indices and the distribution of iceberg trade costs:

   $$(P^{nj})' = \left(\sum_{i=1}^{N} ((x^{ij})^r)^{-\theta^j} (\kappa^{nj,ij})^{-\theta^j} ((A^{ij})^s)^{\theta^j} \right)^{\frac{-1}{\theta^j}}$$

   - Update a new vector of price indices using the previous simulation:

   $$(P^{nj})^{r+1} = \nu (P^{nj})' + (1-\nu)(P^{nj})^r,$$

   where the parameter ν corresponds to a convergence factor.

   - Continue the algorithm until the minimum difference between simulation r and $r+1$ of price index P^{nj} is lower than some tolerance factor. That is: $|(P^{nj})^{r+1} - (P^{nj})^r| < tol$ for all nj.

6. With the updated vector of price indices P^{nj} construct the variable x^{nj}.

7. Construct a variable called $(\tilde{\pi}^{ij,nj})^s$ using the variables x^{ij}, A^{ij}, and the trade costs, $\kappa^{ij,nj}$:

$$\tilde{\pi}^{ij,nj} = \frac{(x^{nj})^{-\theta^j}(\kappa^{ij,nj})^{-\theta^j}}{\sum_m (x^{mj})^{-\theta^j}(\kappa^{ij,mj})^{-\theta^j}((A^{m,j})^s)^{\theta^j}}$$

8. Construct a new vector of productivity $(A^{nj})'$ using $\tilde{\pi}^{ij,nj}$ and the initial data Y^{nj}:

$$(A^{nj})' = \left(\frac{Y^{nj}}{\sum_{i=1}^{N}(\tilde{\pi}^{ij,nj})^s}\right)^{\frac{1}{\theta^j}}$$

9. Update a new vector of productivity with some convergence factor ζ and the productivity vector of simulation s:

$$(A^{nj})^{s+1} = \zeta(A^{nj})' + (1-\zeta)(A^{nj})^s$$

10. Continue the algorithm until the minimum difference between simulation s and $s+1$ of productivity A^{nj} is lower than some tolerance factor. That is: $|(A^{nj})^{s+1} - (A^{nj})^s| < tol$ for all nj.

C Existence and Uniqueness of the Equilibrium

In this section, we examine the conditions under which the *steady-state* equilibrium exists and is unique, following Allen et al. (2024) and Kleinman et al. (2023). The model incorporates multiple sectors, with agglomeration forces assumed to operate only in the urban sector. To avoid equilibrium multiplicity in the simulations, we set the strength of agglomeration forces to a very small value. Specifically, we choose a value of 0.05, ensuring that agglomeration effects remain weaker than congestion forces.[2] Nevertheless, in the full multi-sector model, it is difficult to derive general conditions for uniqueness and existence, since the theorem of Allen et al. (2024) cannot be directly applied to a dynamic model with multiple sectors. To address this limitation, in this section, we restrict attention to a one-sector version of the economy, which allows us to establish conditions for uniqueness in the presence of agglomeration forces in a particular location. Agglomeration forces take the following form:

$$A^i_* = \tilde{A}^i(L^i_*)^\gamma,$$

where γ corresponds to the strength of the agglomeration force and $\tilde{A}_i$ to a scale productivity parameter that in the steady state remains constant. Assuming time-invariant fundamentals: $\{\tilde{A}^i, K^i, \kappa^{n,i}, f^{n,i}\}$, endogenous variables: $\{w^i_*, P^i_*, v^i_*, r^i_*, l^i_*\}$ where x^i_* corresponds to the value for variable x in the deterministic steady state equilibrium and a set of parameters $\{\beta, \nu, \phi, \eta, \theta, \gamma\}$.

[2] To corroborate this very small value, we also simulate the shock from different initial points and make sure that we always reach the same steady state equilibrium for the same shock in the model.

Let's define the following variables:

$$\exp\left(\frac{\beta}{\nu}V_*^i\right) = \left(\frac{w_*^i}{P_*^i}\right)(v_*^i)^{\beta}$$
$$\tilde{f}^{i,n} = \left(\exp(f^{i,n})\right)^{\frac{-1}{\nu}}.$$

Then, we have that the following system of equations must hold in steady state:

Price indices

$$(P_*^i)^{-\theta} = \sum_{n=1}^{N}(\kappa^{i,n}(w_*^n)^{\phi}(P_*^n)^{1-\phi})^{-\theta}(\tilde{A}^n)^{\theta}(L_*^n)^{\theta(\gamma-\phi(1-\eta))}(K^n)^{-\theta\phi(1-\eta)} \tag{C.1}$$

Labor market clearing condition

$$(w_*^i)^{1+\theta\phi}(L_*^i)^{1-\theta(\gamma-\phi(1-\eta))}(P_*^i)^{\theta(1-\phi)}(K^i)^{\theta\phi(1-\eta)} = \eta\sum_{n=1}^{N}\left(\frac{\kappa^{n,i}}{\tilde{A}^i}\right)^{-\theta}(P_*^n)^{\theta}w_*^nL_*^n \tag{C.2}$$

Labor mobility condition

$$(P_*^i)^{\frac{\beta}{\nu}}(w_*^i)^{\frac{-\beta}{\nu}}L_*^i(v_*^i)^{-\beta} = \sum_{n=1}^{N}\tilde{f}^{n,i}L_*^n(v_*^n)^{-1} \tag{C.3}$$

Value function

$$v_*^i = \sum_{n=1}^{N}\tilde{f}^{i,n}(P_*^n)^{\frac{-\beta}{\nu}}(w_*^n)^{\frac{\beta}{\nu}}(v_*^n)^{\beta} \tag{C.4}$$

Then, we can apply the theorem from Allen et al. (2024) for the endogenous variables: $\{w_*^i, P_*^i, L_*^i, v_*^i\}$, which states that a sufficient condition for the existence and uniqueness of a steady state equilibrium is that the spectral radius of matrix A determined by the exponents in the system of equations C.1 to C.4 is less than or equal to one.

The following matrix can represent the exponents on the endogenous variables in the left-hand side:

$$\Lambda = \begin{bmatrix} 0 & -\theta & 0 & 0 \\ 1+\theta\phi & \theta(1-\phi) & 1-\theta(\gamma-\phi(1-\eta)) & 0 \\ \frac{-\beta}{\nu} & \frac{\beta}{\nu} & 1 & -\beta \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and on the right-hand side by:

$$\Gamma = \begin{bmatrix} -\theta\phi & -\theta(1-\phi) & \theta(\gamma-\phi(1-\eta)) & 0 \\ 1 & \theta & 1 & 0 \\ 0 & 0 & 1 & -1 \\ \frac{\beta}{\nu} & \frac{-\beta}{\nu} & 0 & \beta \end{bmatrix}$$

Let $A \equiv |\Gamma\Lambda^{-1}|$ and denote the spectral radius as $\rho(A)$. Following Theorem 1 from Allen et al. (2024), a sufficient condition for existence and uniqueness is that $\rho(A) \leq 1$. Notice that the system will have exponents that will not be very strong agglomeration forces if $\gamma \leq \frac{1}{\theta} + \phi(1-\eta)$, which is a condition that is satisfied in our simulations. It is difficult to solve the spectral radius of this matrix in parametric terms.

Then, to provide the exact condition, we proceed as Kleinman et al. (2023), assuming symmetric trade and migration costs across locations: $\tilde{f}^{i,n} = \tilde{f}^{n,i}$ and $\frac{\kappa^{n,i}K^i}{\tilde{A}^i} = \frac{\kappa^{i,n}K^n}{\tilde{A}^n}$. We can transform equations C.3 and C.4 as follows:

From equation C.4, we get that:

$$1 = \sum_{n=1}^{N} \frac{\tilde{f}^{i,n}(\omega_*^n)^{\frac{\beta}{\nu}}(v_*^n)^\beta}{v_*^i} \tag{C.5}$$

where $\omega_*^n = \frac{w_*^n}{P_*^n}$. Then, we can rewrite equation C.3 as:

$$(\omega_*^i)^{\frac{-\beta}{\nu}} L_*^i (v_*^i)^{-\beta} \left(\sum_{n=1}^{N} \frac{\tilde{f}^{i,n}(\omega_*^n)^{\frac{\beta}{\nu}}(v_*^n)^\beta}{v_*^i} \right) = \sum_{n=1}^{N} \tilde{f}^{n,i} L_*^n (v_*^n)^{-1} \tag{C.6}$$

Rearranging terms

$$\sum_{n=1}^{N} \frac{\tilde{f}^{i,n}(\omega_*^n)^{\frac{\beta}{\nu}}(v_*^n)^\beta}{v_*^i} L_*^i = \sum_{n=1}^{N} \frac{\tilde{f}^{n,i}(\omega_*^i)^{\frac{\beta}{\nu}}(v_*^i)^\beta}{v_*^n} L_*^n \tag{C.7}$$

From the symmetric condition and also rearranging some variables, we get that:

$$\sum_{n=1}^{N} \tilde{f}^{i,n}(\omega_*^n)^{\frac{\beta}{\nu}}(v_*^n)^\beta \frac{L_*^i}{v_*^i} = \sum_{n=1}^{N} \tilde{f}^{i,n} \frac{L_*^n}{v_*^n}(\omega_*^i)^{\frac{\beta}{\nu}}(v_*^i)^\beta \tag{C.8}$$

Then, applying the Perron-Frobenius Theorem, we have that $\frac{L_*^i}{v_*^i} = \chi(\omega_*^i)^{\frac{\beta}{\nu}}(v_*^i)^\beta$ for some constant χ and w.l.o.g, we can assume $\chi = 1$. Thus, we can use the following equation to reduce the system by one endogenous variable:

$$L_*^i = (\omega_*^i)^{\frac{\beta}{\nu}}(v_*^i)^{1+\beta} \tag{C.9}$$

We then can follow a similar approach for equations C.1 and C.2. Let's define $\varsigma = \gamma - \phi(1-\eta)$ From equation C.1 and using equation C.9, we get that:

$$1 = \sum_{n=1}^{N} (\kappa^{i,n}(w_*^n)^\phi (P_*^n)^{1-\phi})^{-\theta} (\tilde{A}^n)^\theta ((\omega_*^i)^{\frac{\beta}{\nu}}(v_*^i)^{1+\beta})^{\theta\varsigma} (K^n)^{\theta\phi(1-\eta)} (P_*^i)^\theta \tag{C.10}$$

Multiplying the L.H.S of equation C.2 by this term and rearranging some variables, we obtain:

$$\sum_{n=1}^{N} (K^n)^{-\theta\phi(1-\eta)} (\kappa^{i,n}(w_*^n)^\phi (P_*^n)^{1-\phi})^{-\theta} (\tilde{A}^n)^\theta (L_*^n)^{\theta\varsigma} (P_*^i)^\theta w_*^i L_*^i \tag{C.11}$$

$$= \eta \sum_{n=1}^{N} \left(\frac{\kappa^{n,i}}{\tilde{A}^i} \right)^{-\theta} (K^i)^{-\theta\phi(1-\eta)} (P_*^n)^{\theta} w_*^n L_*^n (P_*^i)^{-\theta(1-\phi)} (w_*^i)^{-\theta\phi} (L_*^i)^{\theta\varsigma} \tag{C.12}$$

Replacing $\frac{w_*^i}{P_*^i}$ in terms of ω_*^i, we get that:

$$\sum_{n=1}^{N} (K^n)^{-\theta\phi(1-\eta)} (\kappa^{i,n} (\omega_*^n)^{\phi} (P_*^n))^{-\theta} (\tilde{A}^n)^{\theta} (L_*^n)^{\theta\varsigma} (P_*^i)^{1-\theta} \omega_*^i L_*^i \tag{C.13}$$

$$= \eta \sum_{n=1}^{N} \left(\frac{\kappa^{n,i}}{\tilde{A}^i} \right)^{-\theta} (K^i)^{-\theta\phi(1-\eta)} (P_*^n)^{1-\theta} \omega_*^n L_*^n (P_*^i)^{-\theta} (\omega_*^i)^{-\theta\phi} (L_*^i)^{\theta\varsigma} \tag{C.14}$$

Since trade costs are symmetric, we can apply the Perron-Frobenius theorem again, we have that:

$$(P_*^i)^{1-\theta} \omega_i^* L_i^* = \zeta (P_*^i)^{-\theta} (\omega_*^i)^{-\theta\phi} (L_*^i)^{\theta\varsigma}$$

for some constant ζ that w.l.o.g we can assume $\zeta = 1$. This implies that we can use the following equation to solve for the price index:

$$P_*^i = (\omega_*^i)^{-\theta\phi-1} (L_*^i)^{\theta\varsigma-1} \tag{C.15}$$

Replacing the employment from equation C.9, we get that:

$$P_*^i = (\omega_*^i)^{\frac{\beta}{\nu}(\theta\varsigma-1)-\theta\phi-1} (v_*^i)^{(1+\beta)(\theta\varsigma-1)} \tag{C.16}$$

Then we get the following system of equations of two endogenous variables v_*^i and ω_*^i:

Value function equations:

$$v_i^* = \sum_{n=1}^{N} \tilde{f}^{\,i,n} \, (\omega_n^*)^{\frac{\beta}{\nu}} \, (v_n^*)^{\beta}, \qquad i = 1, \ldots, N. \tag{C.17}$$

Price index equations:

$$(\omega_i^*)^{\theta\left[\frac{\beta}{\nu}(1-\theta\varsigma)+\theta\phi+1\right]} \, (v_i^*)^{(1+\beta)\theta(1-\theta\varsigma)} \tag{C.18}$$

$$= \sum_{n=1}^{N} (\kappa^{i,n})^{-\theta} \, (\tilde{A}^{\,n})^{\theta} \, (K^n)^{\theta\phi(1-\eta)} \, (\omega_n^*)^{\theta\left\{\frac{\beta}{\nu}[\varsigma(\theta+1)-1]-(\phi(\theta+1)+1)\right\}} \, (v_n^*)^{(1+\beta)\theta[\varsigma(\theta+1)-1]}. \tag{C.19}$$

Then, we can apply again Theorem 1 from Allen et al. (2024) to look for the conditions of existence and uniqueness. In this case:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ (1+\beta)\theta(1-\theta\varsigma) & \theta\left[\frac{\beta}{\nu}(1-\theta\varsigma)+\theta\phi+1\right] \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} \beta & \frac{\beta}{\nu} \\ (1+\beta)\theta\left[\varsigma(\theta+1)-1\right] & \theta\left\{\frac{\beta}{\nu}\left[\varsigma(\theta+1)-1\right]-(\phi(\theta+1)+1)\right\} \end{bmatrix}$$

Then we have that $A = |\Gamma\Lambda^{-1}|$, and we can try to look for the conditions such that $\rho(A) \leq 1$. Defining the following parameters:

$$\varsigma \equiv \gamma - \phi(1-\eta), \qquad \Delta \equiv 1 - \theta\varsigma, \qquad \Sigma \equiv 1 - (\theta+1)\varsigma,$$

$$E \equiv \frac{\beta}{\nu}\Delta + \theta\phi + 1, \qquad D \equiv \theta E, \qquad R \equiv \phi(\theta+1) + 1 + \frac{\beta}{\nu}\Sigma,$$

the inverse of the exponent matrix is

$$\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{(1+\beta)\Delta}{E} & \frac{1}{D} \end{bmatrix},$$

and the right-hand side matrix is

$$\Gamma = \begin{bmatrix} \beta & \frac{\beta}{\nu} \\ (1+\beta)\theta[\varsigma(\theta+1)-1] & \theta\left\{\frac{\beta}{\nu}[\varsigma(\theta+1)-1]-(\phi(\theta+1)+1)\right\} \end{bmatrix}.$$

Taking the product and the absolute value, we obtain

$$A \equiv \left|\Gamma\Lambda^{-1}\right| = \begin{bmatrix} \left|\beta - \frac{\beta}{\nu}\frac{(1+\beta)\Delta}{E}\right| & \frac{\beta/\nu}{\theta E} \\ \theta(1+\beta)\left|\frac{R\Delta}{E} - \Sigma\right| & \frac{R}{E} \end{bmatrix}.$$

We then check that the spectral radius, $\rho(A) \leq 1$, in our simulations.

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